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Identification of the Optimal Active Set in a Noninterior Continuation Method for LCP

NAIHUA XIU[1] and JIANZHONG ZHANG
[1]*Department of Applied Mathematics, Northern Jiaotong University, Beijing 100044, China; E-mail: nhxiu@center.njtu.edu.cn;* [2] *Department of Mathematics, City University of Hong Kong, Hong Kong, China; E-mail: mazhang@cityu.edu.hk*

Abstract. This paper concerns about the possibility of identifying the active set in a noninterior continuation method for solving the standard linear complementarity problem based on the algorithm and theory presented by Burke and Xu (J. Optim. Theory Appl. 112 (2002) 53). It is shown that under the assumptions of P-matrix and nondegeneracy, the algorithm requires at most $O\,(\rho \log\,(\beta_0\mu_0/\tau))$ iterations to find the optimal active set, where β_0 is the width of the neighborhood which depends on the initial point, $\mu_0 > 0$ is the initial smoothing parameter, ρ is a positive number which depends on the problem and the initial point, and τ is a small positive number which depends only on the problem.

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1. Introduction

Let $M \in R^{n\times n}$ be a given matrix and $q \in R^n$ be a given vector. The standard linear complementarity problem, which is denoted by LCP(M, q), is to find a vector $(x, y) \in R^{2n}$ such that

$$\begin{cases} Mx + q - y = 0, \\ x^T y = 0,\ x \geqslant 0,\ y \geqslant 0. \end{cases} \tag{1.1}$$

Various well-known methods have been proposed to solve LCP(M, q), see the book by Cottle, Pang and Stone [14] and the survey by Ferris and Kanzow [19]. Among them, smoothing (Newton) methods have received an increasing interest in the literature for solving complementarity and variational inequality problems, see e.g., [1–13,18,21–24,26–27,30–34,36–39], the survey [35] and the numerical report [40] for the details

The main feature of smoothing methods is to construct a smooth approximation to a nonsmooth equation reformulation of the concerned problem, and then use the Newton method to solve the smoothing equation. These methods have global convergence and locally superlinear/quadratic convergence under certain conditions. However, a general smoothing method for solving (1.1) does not possess a finite convergence property since the smoothing equation is always nonlinear. Usually, the convergence behavior of an algorithm is closely related to the correct identification of the optimal active constraints. For example, in the field of interior point methods, the identification technique, after finitely many iterations, allows us to recover an exact solution easily from the approximate one which is provided by an interior point method. Such an identification property improves the efficiency of interior point methods and column generation techniques. For the related discussion, see, e.g. [15–17,25] and the references therein. In this sense, the study of the identification property is an interesting topic in the field of smoothing methods. This paper will focus on such an identification property and its complexity bounds for a noninterior continuation method which is viewed as a special smoothing method.

Complexity of the smoothing methods for finding an ϵ-approximate solution to problem (1.1) has been studied by Hotta et al. [24] and Burke and Xu [3]. In [24], the authors obtain a complexity bound under the assumption of monotonicity. In [3], the authors show the complexity bound under the assumption of P-matrix.

Motivated by the algorithm and theory of Burke and Xu [3], we first give a minor modification of their algorithm in Section 2 of this paper. We then show in Section 3 that, under the assumptions of P-matrix and nondegeneracy, the modified algorithm can identify the *optimal active set* if the starting point is in a small and narrow neighborhood of the central path. By using this result, we finally show in Section 4 that the modified algorithm with arbitrary starting point finds the *optimal active set* by at most $O\left(\rho \log\ (\beta_0\mu_0/\tau)\right)$ iterations, where β_0 is the width of the neighborhood which depends on the initial point, μ_0 is the initial smoothing parameter, ρ is a positive number which depends on the problem and the initial point, and τ is a small positive number which depends only on the problem. Once the optimal face is identified, the exact solution of problem (1.1) can be obtained immediately by solving a linear system.

The following notation will be used throughout the paper. R^n_+ and R^n_{++} denote the nonnegative and positive orthants of R^n, respectively. $\|\cdot\|$ represents 2-norm, while other norms will be expressed with appropriate subscripts. In addition, $vec\{x_i\} = x$. All vectors are column vectors. For simplicity, we sometimes use (x, y) for the column vector $(x^T, y^T)^T$. Matrix E represents the identity matrix with suitable dimension. Given matrix $W \in R^{n\times n}$, and $J, K \subseteq I := \{1, 2, \cdots, n\}$, we denote by W_{JK} the $|J| \times |K|$ submatrix of W consisting of entries w_{jk}, $j \in J, k \in K$.

2. A modified algorithm

The algorithm proposed in this paper is based on the use of the Chen-Harker-Kanzow-Smale (CHKS) smoothing function

$$p(u,\mu) = \frac{u + \sqrt{u^2 + 4\mu^2}}{2},$$

where $\mu > 0$ is called the smoothing parameter. Chen and Harker [6] used this function to construct the first noninterior continuation method for the LCP. Several properties of this function have been observed by Burke and Xu [2], Hotta and Yoshise [23], Kanzow [26], Qi and Sun [33–35], etc.

LEMMA 2.1. *Let $p(u,\mu)$ be the CHKS smoothing function. Then $p(u,\mu)$ is twice continuously differentiable on $R \times R_{++}$. Moreover,*
(i) $0 < p_1'(u,\mu) < 1,\ p_1'(-u,\mu) = 1 - p_1'(u,\mu),\ \forall u \in R, \mu > 0$;
(ii) $0 < p_2'(u,\mu) \leqslant \min\{1, 2\mu/|u|\},\ \forall u \in R,\ \mu > 0$;
(iii) $0 < p_{ii}''(u,\mu) \leqslant \min\{1/|u|, 1/\mu\},\ \forall u \in R,\ \mu > 0$, *i=1,2;*
(iv) $0 < p(u,\mu) - p(u,0) \leqslant 2\mu^2/|u|,\ \forall u \in R,\ \mu > 0.$

Let $(\cdot)_+$ denote the componentwise maximum, then the problem (1.1) is reformulated as

$$F(w) := \begin{bmatrix} Mx + q - y \\ x - (x-y)_+ \end{bmatrix} = 0, \tag{2.1}$$

where $w = (x, y) \in R^{2n}$. This is a system of nonsmooth equations. By using the CHKS smoothing function, we can construct a smooth approximation to (2.1):

$$H(w,\mu) := \begin{bmatrix} Mx + q - y \\ \Phi(w,\mu) \end{bmatrix} = 0, \tag{2.2}$$

where $\mu > 0$, $\Phi(w,\mu) = (\phi(x_1, y_1, \mu), \cdots, \phi(x_n, y_n, \mu))^T$, and

$$\phi(r,s,\mu) = r - p(r-s,\mu), \quad \forall (r,s) \in R^2.$$

It is easy to check that the Jacobian matrix of H with respect to w has the following form:

$$\nabla_w H(w,\mu) = \begin{bmatrix} M & -E \\ \nabla_x \Phi(w,\mu) & \nabla_y \Phi(w,\mu) \end{bmatrix},$$

where

$$\nabla_x \Phi(w,\mu) = diag\{1 - p_1'(x_i - y_i, \mu)\} = \frac{1}{2}\, diag \left\{ 1 - \frac{x_i - y_i}{\sqrt{(x_i - y_i)^2 + 4\mu^2}} \right\},$$

and

$$\nabla_y \Phi(w,\mu) = diag\{p_1'(x_i - y_i, \mu)\} = \frac{1}{2} diag\left\{1 + \frac{x_i - y_i}{\sqrt{(x_i - y_i)^2 + 4\mu^2}}\right\}.$$

We now describe our algorithm in which we use a concept of neighborhood of the central path that was introduced by Burke and Xu [1], i.e.,

$$\mathcal{N}(\beta,\mu) := \{w = (x, y) \in R^{2n} \mid Mx + q = y, \|\Phi(w,\mu)\|_\infty \leqslant \beta\mu\},$$

where parameter $\beta > 0$ is called the width of the neighborhood. For a suitable initial point $(w^0, \mu_0) \in R^{2n} \times R_{++}$, we take $\beta_0 \geqslant \|\Phi^0\|_\infty/\mu_0$ where $\Phi^0 = \Phi(w^0, \mu_0)$. If $\beta_0 \leqslant \beta^*$ (a small positive number, see Remark 1 below), then let $\beta = \beta_0$ and we directly use Burke-Xu's algorithmic framework to produce an iterative sequence $\{w^k, \mu_k\}$. If $\beta_0 > \beta^*$, then we apply the damped Newton method to equation $H(w, \mu_0) = 0$ to produce an iterative sequence $\{w^{0,l}\}$ until $\|\Phi^{0,l}\|_\infty/\mu_0 \leqslant \beta^*$ is satisfied, and let β be a positive number less than β^*. The details of the modified algorithm are stated as follows. For simplicity, at the kth iteration we use $H^k = H(w^k, \mu_k)$, $\Phi^k = \Phi(w^k, \mu_k)$, $\nabla_w H^k = \nabla_w H(w^k, \mu_k)$, $H^{0,l} = H(w^{0,l}, \mu_0)$, $\Phi^{0,l} = \Phi(w^{0,l}, \mu_0)$, etc.

Algorithm: Given constant $\sigma \in (0, 1/2]$.

Step 0. (Preliminary)

Take $x^0 \in R^n$, $\mu_0 > 0$ and $y^0 = Mx^0 + q$. Let $\beta_0 \geqslant \max\{1, \|\Phi^0\|_\infty/\mu_0\}$.

S0.1. Let $l := 0$, $x^{0,l} := x^0$, $y^{0,l} := y^0$.

S0.2. If $\|\Phi^{0,l}\|_\infty/\mu_0 \leqslant \beta^*$ (see Remark 1 below), then set $x^0 := x^{0,l}$, $y^0 := y^{0,l}$, and $\beta := (1-\sigma)\beta^*$ if $\|\Phi^{0,l}\|_\infty/\mu_0 < (1-\sigma)\beta^*$, and $\beta := \|\Phi^{0,l}\|_\infty/\mu_0$ otherwise. Set $k := 0$, and go to Step 1.

S0.3. Set $w^{0,l+1} = w^{0,l} + \theta_{0,l}\triangle w^{0,l}$, where $\triangle w^{0,l} = (\triangle x^{0,l}, \triangle y^{0,l})$ satisfies the linear system

$$\nabla_w H(w^{0,l}, \mu_0)\triangle w = -H(w^{0,l}, \mu_0), \tag{2.3}$$

and $\theta_{0,l}$ is the maximum in the set $\{1, 1/2, 1/4, \cdots\}$ such that

$$\|\Phi(w^{0,l+1}, \mu_0)\|_\infty \leqslant (1 - \sigma\theta_{0,l})\|\Phi(w^{0,l}, \mu_0)\|_\infty. \tag{2.4}$$

S0.4. Let $l := l + 1$ and go to Step S0.2.

Step 1. If $\|\Phi^k\|_\infty = 0$ then stop; otherwise go to the next step.

Step 2. (The Search Direction)

Compute the Newton direction $\triangle w^k = (\triangle x^k, \triangle y^k)$ by solving the linear system

$$\nabla_w H(w^k, \mu_k)\triangle w = -H^k. \tag{2.5}$$

Step 3. (The New Iterative Point)

Set $w^{k+1} = w^k + \theta_k\triangle w^k$, where θ_k is the maximum in the set $\{1, 1/2, 1/4, \cdots\}$

such that

$$\|\Phi(w^{k+1}, \mu_k)\|_\infty \leqslant (1 - \sigma\theta_k)\|\Phi(w^k, \mu_k)\|_\infty. \tag{2.6}$$

Step 4. (Update for μ_k)

$$\mu_{k+1} = \left(1 - \frac{1}{2}\beta\sigma\theta_k\right)\mu_k. \tag{2.7}$$

Step 5. Set $k := k + 1$, and go to Step 1.

REMARKS

(1) In the algorithm, the choice of the starting point (w^0, μ_0) is quite easy and arbitrary. Steps S0.1-S0.4 are called preliminary iterations or Phase (I). They are added to ensure that the width of the neighborhood is very small, i.e., $w^0 \in \mathcal{N}(\beta, \mu_0)$ and

$$(1 - \sigma)\beta^* \leqslant \beta \leqslant \beta^*, \quad \beta^* := \frac{\mathcal{L}}{2(1 + \mathcal{L})} < 1, \tag{2.8}$$

where $\mathcal{L}$ is the fundamental quantity associated with P-matrix (see Section 3 for details). We will see from the analysis below that β^* can be replaced by one of its lower bounds.

(2) From Kanzow [26], the Newton equations (2.3) and (2.5) are both solvable under the assumption that M is a P_0-matrix.

(3) By applying the same proof techniques used in Burke and Xu [1–3], we can show that the backtracking line search procedure for evaluating θ_k in Step 3 ($\theta_{0,l}$ in Step S0.3) is finitely terminating, and that μ_{k+1} is well-defined and $w^{k+1} \in \mathcal{N}(\beta, \mu_{k+1})$.

The above remarks imply that the modified algorithm is implementable. If it produces an infinite sequence $\{w^k, \mu_k\}$, we can derive the following global linear convergence theorem, whose proof is similar to the one in Burke and [1] or Chen and [8], and hence is omitted.

THEOREM 2.1. *Suppose that M is a P-matrix and (x^*, y^*) is the unique solution to problem (1.1). Let $\{w^k, \mu_k\}$ be an infinite sequence generated by the modified algorithm, then*

(a) *the sequence $\{\mu_k\}$ converges Q-linearly to zero;*

(b) *the sequence $\{(x^k, y^k)\}$ converges R-linearly to the unique solution (x^*, y^*).*

3. Identification of the optimal active set

For $w \in R^{2n}$, we define the index sets

$$\begin{aligned} \mathcal{A}(w) &= \{i \in I \mid x_i - y_i < 0\}, \\ \mathcal{B}(w) &= \{i \in I \mid x_i - y_i > 0\}, \\ \mathcal{C}(w) &= \{i \in I \mid x_i - y_i = 0\}, \end{aligned}$$

where I is given at the end of Section 1. Obviously, when the solution (x^*, y^*) to problem (1.1) is strictly complementary, $\mathcal{A}(w^*) \cup \mathcal{B}(w^*) = I$ (or $\mathcal{C}(w^*) = \emptyset$), and $\mathcal{A}(w^*)$ is the optimal active set defined by $\{i \in I \mid x_i^* = 0\}$. In this section we are to show that if a starting point $w^0 \in R^{2n}$ is given in a small and narrow neighborhood of the central path, then the optimal active set can be identified by an index set at w^k for each k. For this purpose, we make use of the following assumptions.

ASSUMPTION (A) M is a P-matrix.

ASSUMPTION (B) The solution (x^*, y^*) is nondegenerate or strictly complementary.

A matrix $M \in R^{n \times n}$ is said to be a P-matrix if for any nonzero vector $x \in R^n$, $\exists i \in I$: $x_i(Mx)_i > 0$. It is known that if M is a P-matrix, then problem (1.1) has a unique solution, say (x^*, y^*). In [29], Mathias and Pang proved that, if M is a P-matrix, then there is a constant $\alpha(M) := \min_{\|x\|_\infty = 1}\{\max_i x_i(Mx)_i\} > 0$ such that

$$\max_i \; x_i(Mx)_i \geqslant \alpha(M)\|x\|_\infty^2, \quad \forall x \in R^n, \tag{3.1}$$

and

$$\|M\|_\infty \geqslant \alpha(M), \quad \alpha(M)\alpha(M^{-1}) \leqslant 1. \tag{3.2}$$

Furthermore, [28] gave a lower bound estimate for $\alpha(M)$.

Let $\mathcal{P}_J(M)$ be the principal pivotal transform of M with respect to the index sets $J \subseteq I$ and $\bar{J} := I \backslash J$ defined by

$$\mathcal{P}_J(M) := \begin{bmatrix} M_{JJ}^{-1} & -M_{JJ}^{-1} M_{J\bar{J}} \\ M_{\bar{J}J} M_{JJ}^{-1} & M_{\bar{J}\bar{J}} - M_{\bar{J}J} M_{JJ}^{-1} M_{J\bar{J}} \end{bmatrix}.$$

Burke and Xu [3] introduced another fundamental quantity associated with a P-matrix M

$$\mathcal{L} := \min\{\alpha(\mathcal{P}_J(M)) \mid \forall J \subseteq I\}, \tag{3.3}$$

which is a finite number and in the interval $(0, \alpha(M))$. Furthermore, they gave a lower bound estimate of such a quantity, see [39] and Theorem 5.3 in [3]. By using the value $\mathcal{L}$, they obtained the following important result in the analysis of complexity.

LEMMA 3.1. (3, Lemma 4.4). *Suppose that Assumption (A) holds. Let* $f(w,\mu)$ *be a continuous function from* $R^{2n} \times R_{++}$ *to* R^n. *Then for any given* $w \in R^{2n}$ *and* $\mu > 0$, *the linear equation*

$$\nabla_w H(w,\mu)\begin{bmatrix} d^1 \\ d^2 \end{bmatrix} = \begin{bmatrix} 0 \\ f(w,\mu) \end{bmatrix} \tag{3.4}$$

is solvable, and the solution (d^1, d^2) *possesses the following bound:*

$$\|(d^1, d^2)\|_\infty \leqslant \left(1 + \frac{1}{\mathcal{L}}\right)\|f(w,\mu)\|_\infty. \tag{3.5}$$

We now consider the condition number of problem (1.1)

$$\sigma_{LCP} := \min_i |x_i^* - y_i^*|, \tag{3.6}$$

which is well-defined, finite and positive when Assumptions (A) and (B) are satisfied. It is the same as the first condition number defined by Illés, Peng et al. [25]. Thus by Lemma 3.2 in [3], there is a cheap way to get a lower bound for σ_{LCP} if the problem data (M, q) are integer.

LEMMA 3.2. *If* M *and* q *are integral, then* $\sigma_{LCP} \geqslant \frac{1}{\pi(M)}$, *where* $\pi(M) := \prod_{j=1}^n \|M_{.j}\|$ *and* $M_{.j}$ *denotes the* j*th column of* M. *That is,* σ_{LCP} *can be bounded by an expression which is formed by the input data.*

Consequently, we obtain a lemma that plays a key role in deriving the desired result.

LEMMA 3.3. *Suppose that Assumptions (A) and (B) hold. If a point* $w^0 \in R^{2n}$ *and a smoothing parameter* $\mu_0 > 0$ *satisfy*

$$\begin{cases} \beta \in (0, \beta^*], \quad \beta^* = \dfrac{\mathcal{L}}{2(1+\mathcal{L})}, & (3.7a) \\ w^0 \in \mathcal{N}(\beta, \mu_0), & (3.7b) \\ \mu_0 \leqslant \dfrac{1}{2}\sigma\beta \cdot \min_i |x_i^0 - y_i^0|, & (3.7c) \end{cases}$$

then $\mathcal{A}(w^0) = \mathcal{A}(w^*)$ and $\mathcal{B}(w^0) = \mathcal{B}(w^*)$.

Proof. For every $\mu \in (0, \mu_0]$, consider the nonlinear system

$$H(w,\mu) - \begin{bmatrix} 0 \\ \frac{\mu}{\mu_0}\Phi(w^0, \mu_0) \end{bmatrix} = 0, \quad w \in R^{2n}. \tag{3.8}$$

Since M is a P-matrix, the system (3.8) for every $\mu \in (0, \mu_0]$ has a unique solution, say $w(\mu) = (x(\mu), y(\mu))$. By the implicit function theorem, $w(\mu)$ forms a continuously differentiable path on $(0, \mu_0]$ with $w(\mu_0) = w^0$. Using an argument similar to Theorem 2.1, this path can be extended continuously to $\mu = 0$ by setting $w(0) = w^*$. It is not difficult to verify by (3.8) that as $w^0 \in \mathcal{N}(\beta, \mu_0)$, $w(\mu) \in \mathcal{N}(\beta, \mu)$ for all $\mu \in [0, \mu_0]$. Also, we can prove that for each $\mu \in (0, \mu_0)$,

$$\mu < \frac{1}{2}\sigma\beta \cdot \min_i |x(\mu)_i - y(\mu)_i|. \tag{3.9}$$

In fact, by the mean value and implicit function theorems we have for $\mu = (1 - \sigma\lambda)\mu_0$ with $\lambda \in (0, 1/\sigma)$,

$$w(\mu) = w(\mu_0) + (\mu - \mu_0)d = w^0 - \sigma\lambda\mu_0 d,$$

where $d = (d^1, d^2)$ is defined by

$$d := -(\nabla_w H(\bar{w}, \bar{\mu}))^{-1} \begin{bmatrix} 0 \\ \nabla_\mu \Phi(\bar{w}, \bar{\mu}) - \frac{1}{\mu_0}\Phi(w^0, \mu^0) \end{bmatrix} \tag{3.10}$$

with $\bar{\mu} = (\bar{\mu}_1, \bar{\mu}_2, \cdots, \bar{\mu}_n)^T \in R^n$, $\bar{\mu}_j \in (\mu, \mu_0)$ for every j, and $\bar{w} = w(\bar{\mu}) \in R^{2n}$ (Note that in vectors $x(\mu)$ and $y(\mu)$, only one is independent since $y(\mu) = Mx(\mu) + q$). Then

$$x(\mu) - y(\mu) = (x^0 - y^0) - \sigma\lambda\mu_0(d^1 - d^2),$$

which implies that

$$\min_i \ |x_i^0 - y_i^0| \leqslant \min_i \ |x(\mu)_i - y(\mu)_i| + \sigma\lambda\mu_0\|d^1 - d^2\|_\infty. \tag{3.11}$$

By part (ii) of Lemma 2.1, we have

$$\|\nabla_\mu \Phi(\bar{w}, \bar{\mu}) - \frac{1}{\mu_0}\Phi(w^0, \mu_0)\|_\infty \leqslant 1 + \frac{\|\Phi(w^0, \mu_0)\|_\infty}{\mu_0} \leqslant 1 + \beta < 2. \tag{3.12}$$

Thus, from (3.7c), (3.11), Lemma 3.1, (3.12) and (2.8) we know that

$$\begin{aligned} \mu_0 &\leqslant \frac{1}{2}\sigma\beta \min_i \ |x_i^0 - y_i^0| \\ &\leqslant \frac{1}{2}\sigma\beta[\min_i \ |x(\mu)_i - y(\mu)_i| + \sigma\lambda\mu_0\|d^1 - d^2\|_\infty] \\ &\leqslant \frac{1}{2}\sigma\beta[\min_i \ |x(\mu)_i - y(\mu)_i| + \sigma\lambda\mu_0\frac{2(1+\mathcal{L})}{\mathcal{L}}\|\nabla_\mu\Phi(\bar{w}, \bar{\mu}) \\ &\quad -\frac{1}{\mu_0}\Phi(w^0, \mu_0)\|_\infty] \\ &< \frac{1}{2}\sigma\beta\left[\min_i \ |x(\mu)_i - y(\mu)_i| + \sigma\lambda\mu_0\frac{4(1+\mathcal{L})}{\mathcal{L}}\right] \\ &\leqslant \frac{1}{2}\sigma\beta \min_i \ |x(\mu)_i - y(\mu)_i| + \sigma^2\lambda\mu_0. \end{aligned}$$

This inequality, together with the facts $\mu = (1 - \sigma\lambda)\mu_0$ and $\sigma \in (0, \frac{1}{2}]$, yield that for $\lambda \in (0, \frac{1}{\sigma})$,

$$\begin{aligned}\mu &= (1-\sigma\lambda)\mu_0 < \tfrac{1-\sigma\lambda}{1-\sigma^2\lambda}\frac{1}{2}\sigma\beta \cdot \min_i \; |x(\mu)_i - y(\mu)_i| \\ &< \frac{1}{2}\sigma\beta \cdot \min_i \; |x(\mu)_i - y(\mu)_i|.\end{aligned}$$

That is, (3.9) holds.

From (3.9) and (3.7c) we know that for all i and all $\mu \in (0, \mu_0]$, $x(\mu)_i - y(\mu)_i \neq 0$. Hence continuity of $w(\mu)$ on $[0, \mu_0]$ and Assumption (B) imply that for each i, the sign of $x(\mu)_i - y(\mu)_i$ is constant and the conclusion is proved. □

We observe that conditions (3.7a)–(3.7c) are independent of any algorithm. They actually show that, for any point belonging to a narrow neighborhood ((3.7a) and (3.7b)) and being not "too distant" from the solution ((3.7c)), one is able to identify correctly the sets $\mathcal{A}(w^*)$ and $\mathcal{B}(w^*)$. In view of this lemma, we derive the main result of this section.

THEOREM 3.1. *(Identification of the Optimal Active Set). Suppose that Assumptions (A) and (B) hold. If we choose the initial point* $w^0 \in R^{2n}$ *and the initial smoothing parameter* $\mu_0 > 0$ *such that condition (3.7) is satisfied, and if* $\{(w^k, \mu_k)\}$ *is a sequence produced by the modified algorithm, then for any index* k, $\mathcal{A}(w^k) = \mathcal{A}(w^*)$ *and* $\mathcal{B}(w^k) = \mathcal{B}(w^*)$.

Proof. We prove this theorem by induction. Assume for some index k,

$$\begin{cases} \beta \in (0, \beta^*], \quad \beta^* = \dfrac{\mathcal{L}}{2(1+\mathcal{L})}, & (3.13a) \\ w^k \in \mathcal{N}(\beta, \mu_k), & (3.13b) \\ \mu_k \leqslant \dfrac{1}{2}\sigma\beta \cdot \min\limits_i |x_i^k - y_i^k|, & (3.13c) \end{cases}$$

By Lemma 3.3, it suffices to prove that they also hold for $k+1$.

It is easy to observe that $w^{k+1} \in \mathcal{N}(\beta, \mu_{k+1})$ by (3.13b) and Remark 3. From

$$x^{k+1} - y^{k+1} = (x^k - y^k) + \theta_k(\triangle x^k - \triangle y^k),$$

we have for every $i \in I$,

$$\begin{aligned}|x_i^k - y_i^k| &\leqslant |x_i^{k+1} - y_i^{k+1}| + \theta_k \cdot \|\triangle x^k - \triangle y^k\|_\infty \\ &\leqslant |x_i^{k+1} - y_i^{k+1}| + \theta_k \frac{2(1+\mathcal{L})}{\mathcal{L}} \|\Phi^k\|_\infty \quad \text{(by Lemma 3.1)} \\ &\leqslant |x_i^{k+1} - y_i^{k+1}| + \theta_k \mu_k. \quad (\text{by } \|\Phi^k\|_\infty \leqslant \beta^*\mu_k)\end{aligned}$$

Hence,

$$\min_i |x_i^k - y_i^k| \leqslant \min_i |x_i^{k+1} - y_i^{k+1}| + \theta_k\mu_k.$$

This together with (3.13c) imply that

$$\begin{aligned}\mu_k &\leqslant \frac{1}{2}\sigma\beta \cdot \min_i |x_i^k - y_i^k| \\ &\leqslant \frac{1}{2}\sigma\beta \left[\min_i |x_i^{k+1} - y_i^{k+1}|\right] + \frac{1}{2}\sigma\beta\theta_k\mu_k.\end{aligned}$$

Therefore,

$$\mu_{k+1} = \left(1 - \frac{1}{2}\sigma\beta\theta_k\right)\mu_k \leqslant \frac{1}{2}\sigma\beta \min_i |x_i^{k+1} - y_i^{k+1}|.$$

This completes the proof. □

4. Complexity of finding the optimal active set

We know from Step 0 of the algorithm in Section 2 that, the choice of the starting point is very arbitrary. Generally speaking, such a starting point does not always satisfy condition (3.7). One problem is whether the modified algorithm is able to produce an iterative point satisfying (3.7). If "Yes", can the required number of iterations be estimated? In this section, we shall answer these two questions. To start, we recall a result about error bound by Mathias and Pang [29].

LEMMA 4.1. (29, Lemma 2). *Let Assumption (A) hold and $\gamma = \alpha(M)$, then for any $x \in R^n$ and $y = Mx + q$,*

$$\|x - x^*\|_\infty \leqslant \frac{(1 + \|M\|_\infty)}{\gamma}\|x - (x - y)_+\|_\infty.$$

By using this lemma, we derive an error bound at point w^k by a linear function of μ_k and $\|\Phi^k\|_\infty$.

LEMMA 4.2. *Under Assumption (A), we have, for any index k,*

$$\|(x^k - y^k) - (x^* - y^*)\|_\infty \leqslant \frac{(1 + \|M\|_\infty)^2}{\gamma}(\|\Phi^k\|_\infty + \mu_k).$$

Proof. For any index k, from $y^k = Mx^k + q$, Lemma 4.1 and part (ii) of Lemma 2.1, we obtain

$$\begin{aligned}&\|(x^k - y^k) - (x^* - y^*)\|_\infty \\ \leqslant\ & (1 + \|M\|_\infty)\|x^k - x^*\|_\infty \\ \leqslant\ & \frac{(1 + \|M\|_\infty)^2}{\gamma}\|x^k - (x^k - y^k)_+\|_\infty \\ \leqslant\ & \frac{(1 + \|M\|_\infty)^2}{\gamma}(\|\Phi^k\|_\infty + \|vec\{(x_i^k - y_i^k)_+ - p(x_i^k - y_i^k, \mu_k)\}\|_\infty) \\ \leqslant\ & \frac{(1 + \|M\|_\infty)^2}{\gamma}(\|\Phi^k\|_\infty + \mu_k).\end{aligned}$$

This completes the proof. □

From the above lemma and Assumption (B), we are able to obtain the following key lemma in our analysis on the complexity issue.

LEMMA 4.3. *Let Assumptions (A) and (B) hold. Then there is an integer K_1 satisfying*

$$\mu_{K_1} < \frac{1}{2}\sigma\beta \min_i \, |x_i^{K_1} - y_i^{K_1}|, \tag{4.1}$$

whenever

$$\mu_{K_1} \leqslant \frac{\sigma}{16} \frac{\mathcal{L}}{(1 + \|M\|_\infty)^2}\sigma_{LCP}. \tag{4.2}$$

Proof. By $\mu_k \downarrow 0$, $\mathcal{L} > 0$ and $\sigma_{LCP} > 0$, there is an integer K_1 such that (4.2) is satisfied. From Lemma 4.2, $\|\Phi^k\|_\infty \leqslant \beta\mu_k$, (4.2), $\beta \leqslant \beta^* < 1$, $\mathcal{L} \leqslant \gamma$ and $\sigma \in (0, \frac{1}{2}]$, we obtain

$$\begin{aligned} \|(x^{K_1} - y^{K_1}) - (x^* - y^*)\|_\infty &\leqslant \frac{(1 + \|M\|_\infty)^2}{\gamma}(\|\Phi^{K_1}\|_\infty + \mu_{K_1}) \\ &\leqslant \frac{(1 + \|M\|_\infty)^2}{\gamma}(1 + \beta)\mu_{K_1} \\ &< \frac{1}{8}\sigma_{LCP}, \end{aligned}$$

from which it is implied that

$$\begin{aligned} \sigma_{LCP} &= \min_i |x_i^* - y_i^*| \\ &\leqslant \min_i |x_i^{K_1} - y_i^{K_1}| + \|(x^{K_1} - y^{K_1}) - (x^* - y^*)\|_\infty \\ &< \min_i |x_i^{K_1} - y_i^{K_1}| + \frac{1}{8}\sigma_{LCP}. \end{aligned}$$

This yields that

$$\sigma_{LCP} < \frac{8}{7} \min_i |x_i^{K_1} - y_i^{K_1}|. \tag{4.3}$$

Thus, from (4.2), (4.3), $\mathcal{L} \leqslant \|M\|_\infty$ and $\sigma \in (0, \frac{1}{2}]$, we know that

$$\begin{aligned} \mu_{K_1} &< \frac{\sigma}{16}\frac{\mathcal{L}}{(1 + \|M\|_\infty)^2}\frac{8}{7}\min_i |x_i^{K_1} - y_i^{K_1}| \\ &= \frac{1}{2}\sigma\beta \min_i \, |x_i^{K_1} - y_i^{K_1}| \frac{\mathcal{L}}{7\beta(1 + \|M\|_\infty)^2} \\ &< \frac{1}{2}\sigma\beta \min_i \, |x_i^{K_1} - y_i^{K_1}|. \end{aligned}$$

The proof is completed. □

In view of (4.1) and $w^{K_1} \in \mathcal{N}(\beta, \mu_{K_1})$, the K_1th iterative point w^{K_1} and the smoothing parameter μ_{K_1} satisfy condition (3.7). Hence, Theorem 3.1 implies that $\mathcal{A}(w^k) = \mathcal{A}(w^*)$ and $\mathcal{B}(w^k) = \mathcal{B}(w^*)$ for all $k \geqslant K_1$, and hence the active set at w^{K_1} coincides with the optimal active set. Thus, one can get the exact solution $w^* = (x^*, y^*)$ with $x^* = (0, x^*_{\mathcal{B}(w^{K_1})})$ and $y^* = (y^*_{\mathcal{A}(w^{K_1})}, 0)$, where $x^*_{\mathcal{B}(w^{K_1})}$ and $y^*_{\mathcal{A}(w^{K_1})}$ satisfy a system of linear equations

$$\begin{aligned} &M_{\mathcal{B}(w^{K_1})\mathcal{B}(w^{K_1})}x_{\mathcal{B}(w^{K_1})} = -q_{\mathcal{B}(w^{K_1})},\\ &y_{\mathcal{A}(w^{K_1})} = M_{\mathcal{A}(w^{K_1})\mathcal{B}(w^{K_1})}x_{\mathcal{B}(w^{K_1})} + q_{\mathcal{A}(w^{K_1})}. \end{aligned} \tag{4.4}$$

In what follows, we give a complexity bound for generating the optimal face. Based on the modified algorithm and the above analysis, we only need to estimate the numbers of iterations for Phases (I) and (II), respectively, where Phase (I) consists of all preliminary iterations in order to meet the condition $\|\Phi^{0,l}\|_\infty/\mu_0 \leqslant \beta^*$, while Phase (II) consists of all iterations in which every index is smaller than K_1.

We now show a complexity bound of Phase (I). Since the initial parameter μ_0 does not need to be updated in Phase (I), we can not directly use the complexity result by Burke and Xu [3]. However, their technique of proof will be used in the proof of the following lemma.

LEMMA 4.4. *Suppose Assumption (A) holds. Then the number of iterations for phase (I) is about*

$$O\left(\beta_0\left(\frac{(1+\mathcal{L})}{\mathcal{L}}\right)^2 \cdot \log\frac{\|\Phi^{0,0}\|_\infty}{\beta^*\mu_0}\right). \tag{4.5}$$

Proof. We begin with proving that for every index l,

$$\theta_{0,l} \geqslant \frac{1}{2}\bar{\theta}, \quad \bar{\theta} := \min\{1, (1-\sigma)/(2\beta_0\left(\frac{1+\mathcal{L}}{\mathcal{L}}\right)^2)\}. \tag{4.6}$$

In fact, for index l, define $w^{0,l}(\theta) = (x^{0,l}(\theta), y^{0,l}(\theta))$ with

$$x^{0,l}(\theta) := x^{0,l} + \theta\triangle x^{0,l}, \quad y^{0,l}(\theta) := y^{0,l} + \theta\triangle y^{0,l}, \quad \theta \in [0, 1].$$

Based on the Taylor expansion and the Newton equation (2.3), we have

$$\begin{aligned} \Phi(w^{0,l}(\theta), \mu_0) &= x^{0,l}(\theta) - vec\{p(x_i^{0,l}(\theta) - y_i^{0,l}(\theta), \mu_0)\}\\ &= (x^{0,l} + \theta\triangle x^{0,l}) - \big[vec\{p(x_i^{0,l} - y_i^{0,l}, \mu_0)\} + \theta\\ &\quad \cdot vec\{p_1'(x_i^{0,l} - y_i^{0,l}, \mu_0)(\triangle x^{0,l} - \triangle y^{0,l})_i\} + \tfrac{1}{2}\theta^2\\ &\quad \cdot vec\{p_{11}''(\bar{x}_i^{0,l} - \bar{y}_i^{0,l}, \mu_0)(\triangle x^{0,l} - \triangle y^{0,l})_i^2\}\big]\\ &= \Phi^{0,l} + \theta\left[\nabla_x\Phi^{0,l}\triangle x^{0,l} + \nabla_y\Phi^{0,l}\triangle y^{0,l}\right] - \tfrac{1}{2}\theta^2 vec\\ &\quad \{p_{11}''(\bar{x}_i^{0,l} - \bar{y}_i^{0,l}, \mu_0)(\triangle x^{0,l} - \triangle y^{0,l})_i^2\}\\ &= (1-\theta)\Phi^{0,l} - \tfrac{1}{2}\theta^2 \cdot vec\{p_{11}''(\bar{x}_i^{0,l} - \bar{y}_i^{0,l}, \mu_0)\\ &\quad (\triangle x^{0,l} - \triangle y^{0,l})_i^2\}, \end{aligned} \tag{4.7}$$

where $(\bar{x}_i^{0,l} - \bar{y}_i^{0,l})$ is between $(x_i^{0,l} - y_i^{0,l})$ and $(x_i^{0,l}(\theta) - y_i^{0,l}(\theta))$ for all i. From (iii) of Lemma 2.1 we obtain

$$\|vec\{p''_{11}(\bar{x}_i^{0,l} - \bar{y}_i^{0,l}, \mu_0)(\triangle x_i^{0,l} - \triangle y_i^{0,l})_i^2\}\|_\infty \leqslant \frac{1}{\mu_0}\|\triangle x^{0,l} - \triangle y^{0,l}\|_\infty^2. \tag{4.8}$$

By the Newton equation (2.3) and Lemma 3.1 we have

$$\|\triangle x^{0,l} - \triangle y^{0,l}\|_\infty \leqslant 2\left(1 + \frac{1}{\mathcal{L}}\right)\|\Phi(w^{0,l}, \mu_0)\|_\infty. \tag{4.9}$$

So, it follows from (4.7)-(4.9) and $\|\Phi^{0,l}\|_\infty \leqslant \beta_0\mu_0$ that

$$\|\Phi(w^{0,l}(\theta), \mu_0)\|_\infty \leqslant [1-\theta+2\beta_0\left(\frac{1+\mathcal{L}}{\mathcal{L}}\right)^2\theta^2]\cdot\|\Phi^{0,l}\|_\infty \leqslant (1-\sigma\theta)\|\Phi^{0,l}\|_\infty$$

whenever

$$\theta \leqslant (1-\sigma)/(2\beta_0\left(\frac{1+\mathcal{L}}{\mathcal{L}}\right)^2).$$

The updating rule for $\theta_{0,l}$ implies (4.6). From (2.4) and (4.6) we obtain that for any index l,

$$\|\Phi(w^{0,l+1}, \mu_0)\|_\infty \leqslant (1-\sigma\theta_{0,l})\|\Phi^{0,l}\|_\infty \leqslant \left(1 - \frac{\sigma(1-\sigma)}{4\beta_0\left(\frac{1+\mathcal{L}}{\mathcal{L}}\right)^2}\right)\|\Phi^{0,l}\|_\infty.$$

To ensure that

$$\|\Phi(w^{0,l}, \mu_0)\|_\infty \leqslant \left(1 - \frac{\sigma(1-\sigma)}{4\beta_0\left(\frac{1+\mathcal{L}}{\mathcal{L}}\right)^2}\right)^l\|\Phi^{0,0}\|_\infty \leqslant \beta^*\mu_0,$$

it suffices if we have

$$l\cdot\log\left(1 - \frac{\sigma(1-\sigma)}{4\beta_0\left(\frac{1+\mathcal{L}}{\mathcal{L}}\right)^2}\right) \leqslant l\cdot\left(-\frac{\sigma(1-\sigma)}{4\beta_0\left(\frac{1+\mathcal{L}}{\mathcal{L}}\right)^2}\right) \leqslant \log\frac{\beta^*\mu_0}{\|\Phi^{0,0}\|_\infty}.$$

Therefore, we know that (4.5) provides an upper bound for complexity of Phase (I). □

Similarly, we can give a complexity bound of Phase (II) in which we obtain μ_{K_1} satisfying (4.2) from μ_0.

LEMMA 4.5. *Suppose that Assumptions (A) and (B) hold. Then the number K_1 of iterations for Phase (II) is bounded by*

$$O\left(\left(\frac{1+\mathcal{L}}{\mathcal{L}}\right)^2\cdot\log\frac{\mu_0}{\frac{\sigma}{16}\frac{\mathcal{L}}{(1+\|M\|_\infty)^2}\sigma_{LCP}}\right). \tag{4.10}$$

By adding (4.5) and (4.10) and noticing the facts that $\beta_0 \geqslant 1$ and $\|\Phi^{0,0}\|_\infty \leqslant \beta_0\mu_0$, we obtain an upper bound for the total number of iterations:

$$O\left(\beta_0\left(\frac{1+\mathcal{L}}{\mathcal{L}}\right)^2 \cdot \log \frac{\beta_0\mu_0}{\frac{\sigma}{16}\frac{\beta^*\mathcal{L}}{(1+\|M\|_\infty)^2}\sigma_{LCP}}\right),$$

which can be simplified as

$$O\left(\rho \cdot \log \frac{\beta_0\mu_0}{\tau}\right), \tag{4.11}$$

where

$$\rho = \beta_0\left(\frac{1+\mathcal{L}}{\mathcal{L}}\right)^2 \quad \text{and} \quad \tau = \frac{\sigma}{16}\frac{\beta^*\mathcal{L}}{(1+\|M\|_\infty)^2}\sigma_{LCP}.$$

From the definitions of $\mathcal{L}$ and β_0, we know that ρ is a positive number which depends on the problem and the starting point. From the definitions of $\mathcal{L}$, β^* and σ_{LCP}, we know that τ is a small positive number which depends only on the problem.

THEOREM 4.1. *(Complexity of Finding the Optimal Active Set) If Assumptions (A) and (B) hold, then the modified noninterior continuation algorithm will generate the optimal active set by at most $O\left(\rho \cdot \log\ \beta_0\mu_0/\tau\right)$ Newton iterations. Moreover, the unique solution w^* of problem (1.1) can be obtained by solving the linear system (4.4).*

Compared with the existing identification results in the literature, Theorem 4.1 is better because it gives the required number of iterations to find the optimal active set of problem (1.1). This identification property may allow us, like in the literature of interior point methods, to reduce computational cost and hence to improve the efficiency of noninterior continuation methods. Further research along this direction would be useful.

From Lemma 3.2, we know that σ_{LCP} can be bounded below by an expression of the input data if the problem data (M, q) are integer. Can a lower bound of the value $\mathcal{L}$ be obtained by a cheap way? This problem is worth studying. The complexity bound (4.11) depends on the assumption of strict complementarity, and it would be desirable if this assumption can be removed in theoretical analysis.

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